One-Dimensional Relativistic Dissipative System with Constant Force and its Quantization

G. López,[1] X. E. López,[2] and H. Hernández[1]

For a relativistic particle under a constant force and a linear velocity dissipation force, a constant of motion is found. Problems are shown for getting the Hamiltonian of this system. Thus, the quantization of this system is carried out through the constant of motion and using the quantization on the velocity variable. The dissipative relativistic quantum bouncer is outlined within this quantization approach.

1. INTRODUCTION

It is known that dynamical systems with dissipation (dissipative systems) represent some difficulties for their right formulation in terms of Lagrangian and Hamiltonian formalisms (López, 1996). Normally this dissipation is included in the dynamical equations in a phenomenological way and through a force which depends on the velocity of the particle. If the Hamiltonian of a dissipative system is found, one proceeds to try to quantize the system. This has been able to do for several nonrelativistic systems (Okubo, 1981; Glauber and Man'ko, 1984; Mijatovic *et al.*, 1985; López and González, 2005), but little is know about relativistic dissipative systems. The special theory of relativity, constructed by Poincarè, Lorentz and Einstein at the beginig of the 20th century, is based on conservative systems, despite its enormous success during the last century (Cho, 2005), one may also be interested in the possible changes on the usual dynamical variables (energy or linear momentum, for example) when dissipation is taken into consideration in a relativistic dynamical system. In this work, a constant of motion for a relativistic particle under a constant force and a linear dissipative force is

[1] Departamento de Física de la Universidad de Guadalajara Apartado Postal 4–13744410 Guadalajara, Jalisco, México; e-mail: gulopez@udgserv.cencar.udg.mx, hhernandez@hotmail.com.

[2] Facultad de Ciencias de la UNAM, Apartado postal 70–348, Coyoacán 04511 México; e-mail: xamaneklopez@hotmail.com.

obtained. The constant of motion at first order in the dissipation parameter and the nonrelativistic limit of the constant of motion are analyzed, and the problems for getting the Hamiltonian of the system are outlined. Finally, the quantization for the relativistic dissipative system, at first order on the dissipation parameter, is carried out through the quantization of the velocity and the constant of motion.

2. RELATIVISTIC CONSTANT OF MOTION

The one-dimensional motion of a particle of mass m at rest under a constant force, f, and a linear dissipation force, $-\alpha v$, is governed by the equation

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) = -(f + \alpha v), \tag{1}$$

where v is the velocity of the particle, α is the dissipation parameter, and c is the speed of light. This equation can be written as the following autonomous dynamical system

$$\frac{dx}{dt} = v \tag{2a}$$

$$\frac{dv}{dt} = -\frac{f}{m}(1 + \beta v)(1 - v^2/c^2)^{3/2}, \tag{2b}$$

where β is the constant defined as $\beta = \alpha/f$. A constant of motion for this system is a function $K = K(x, v)$ which satisfies the following partial differential equation of first order (López, 1999)

$$v\frac{\partial K}{\partial x} - \frac{f}{m}(1 + \beta v)(1 - v^2/c^2)^{3/2}\frac{\partial K}{\partial v} = 0. \tag{3}$$

The general solution of this equation is given by

$$K_\beta(x, v) = G(A(v) + fx), \tag{4}$$

where G is an arbitrary function, and $A(v)$ has been defined as

$$A(v) = m\int\frac{v\, dv}{(1 + \beta v)(1 - v^2/c^2)^{3/2}}. \tag{5}$$

The result of the integration of (5) is

$$A(v) = \begin{cases} \dfrac{(1 - \beta v)mc^2}{\phi(v)} + \dfrac{m\beta c^3}{(1 - \beta^2 c^2)^{3/2}}\arcsin\dfrac{\beta c + v/c}{1 + \beta v} & \text{if } \beta c < 1 \\[3mm] \dfrac{mcv}{\sqrt{1 - v^2/c^2}} + \dfrac{mc^2(1 - 2v/c - 2v^2/c^2)}{3(1 + v/c)\sqrt{1 - v^2/c^2}} & \text{if } \beta c = 1 \\[3mm] \dfrac{(1 - \beta v)mc^2}{\phi(v)} + \dfrac{m\beta c^3}{(\beta^2 c^2 - 1)^{3/2}}\log\dfrac{\psi(v)}{1 + \beta v} & \text{if } \beta c > 1 \end{cases} \tag{6}$$

where the functions $\phi(v)$ and $\psi(v)$ have been defined as

$$\phi(v) = \sqrt{1 - v^2/c^2}(1 - \beta^2 c^2) \tag{7a}$$

and

$$\psi(v) = 2\left(c^2\beta^2 + \beta v + \sqrt{1 - v^2/c^2}\sqrt{c^2\beta^2 - 1}\right). \tag{7b}$$

The function G, appearing on (4), can be determinate through the criterion (López, 1996) of having the usual relativistic energy expression of the associated conservative system for β equal to zero (i.e., the nondissipative case),

$$\lim_{\beta \to 0} K_\beta(x, v) = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + fx, \tag{8}$$

which brings about the result $G = I$ (the identity function).[3] Therefore, the constant of motion for the system (2) can be chosen as

$$K_\beta(x, v) = A(v) + fx. \tag{9}$$

This constant of motion brings about the damping effect on the trajectories in the phase space (x, v). Of course, due to multivalue functions of (6), the value of the constant of motion changes for the trajectories going from the $x > 0$ side to the $x < 0$ side of this space, in order to get the spiral falling down to the origin behavior of the trajectories. So, one may say that (9) represents an "almost everywhere" constant of motion of the system (2), in the sense that the set of points where these changes do occur has zero measurement (Hewitt and Stromberg, 1965), one may call it "local constant of motion."

For weak dissipation, one can also make a Taylor expansion on (5) of the term $(1 + \beta v)^{-1}$ to get the constant of motion of the following form

$$K_\beta(x, v) = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - m\beta\left[\frac{vc^2}{\sqrt{1 - v^2/c^2}} - c^3 \arcsin\frac{v}{c}\right] + fx + \phi(v), \tag{10a}$$

where the function Φ is given by the expression

$$\Phi(v) = \sum_{n=2}^{\infty}(-1)^n \beta^n \left[-\frac{c^2 v^n}{(n - 1)\sqrt{1 - v^2/c^2}} + \frac{nc^2}{n - 1}\int\frac{v^{n-1}dv}{(1 - v^2/c^2)^{3/2}}\right]. \tag{10b}$$

Thus, at first order on the dissipation parameter β, one has

$$K_\beta(x, v) = \gamma mc^2 - m\beta c^3\left[\frac{\gamma v}{c} - \arcsin\frac{v}{c}\right] + fx, \tag{11a}$$

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[3] one could add to (8) the term $-mc^2$ to have the usual energy expression for the nonrelativistic case.

where γ has the usual expression,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{11b}$$

Note that the nonrelativistic limit $(v/c \ll 1)$ must not be taken from the case $c\beta < 1$ or the case $c\beta = 1$ but rather from the case $c\beta > 1$. In this way, subtracting the term of energy at rest, mc^2, on the case $c\beta > 1$, the nonrelativistic constant of motion is given by

$$K_\beta(x, v) = \frac{m}{\beta^2} \left[\beta v - \log(1 + \beta v)\right] + fx. \tag{12}$$

Now, using the known expression (Kobussen, 1979; Leubner, 1981; López and Hernández, 1989) to obtain the Lagrangian from the constant of motion,

$$L(x, v) = v \int \frac{K(x, v)dv}{v^2}, \tag{13}$$

one can get the Lagrangian and generalized linear momentum ($p = \partial L/\partial v$) given by

$$L_\beta(x, v) = B(v) - fx \tag{14a}$$

and

$$p(v) = C(v), \tag{14b}$$

where the functions $B(v)$ and $C(v)$ are given in the appendix. In particular, the Lagrangian and the generalized linear momentum for a relativistic particle with dissipation at first order in the dissipation parameter β as in (11a) are

$$L_1(x, v) = -\frac{mc^2}{\gamma} - m\beta c^3 \left[\frac{v}{2c} \log \frac{\gamma - 1}{\gamma + 1} + \frac{\arcsin\frac{v}{c}}{\gamma} + \frac{v}{c} \log \frac{v}{c}\right] - fx \tag{15a}$$

and

$$p(v) = \gamma m v + \beta m c^2 \left[-\frac{5}{2} + \frac{v\gamma}{c} \arcsin \frac{v}{c} - \frac{3}{2} \log \frac{v}{c}\right]. \tag{15b}$$

Similarly, for the nonrelativistic case with dissipation as in (12), one has

$$L_2(x, v) = \frac{m}{\beta^2}(\beta v - 1) \log(1 + \beta v), \tag{16a}$$

and

$$p(v) = \frac{m}{\beta} \left[\log(1 + \beta v) - \frac{1 - \beta v}{1 + \beta v}\right]. \tag{16b}$$

As one can see from (14b), (A_2), (15b) and (16b), it is not possible to have the inverse relation $v = v(p)$. Therefore, their Hamiltonians are expressed only in

an implicit way through the constants of motion (9), (12) and (11). Thus, the quantization of the system (1) can not be carried out with the standard Shrüdinger equation (Messiah, 1958),

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}(\widehat{x}, \widehat{p})\Psi, \tag{17a}$$

or Heisenberg equation,

$$i\hbar \frac{d\widehat{\xi}}{dt} = |\widehat{\xi}, \widehat{H}| + i\hbar \frac{\partial \widehat{\xi}}{\partial t}, \tag{17b}$$

where $\widehat{\xi}$ is any time depending observable. The Feynman path quantization (Feynman and Hibbs, 1965) is, in principle, possible to use here since one has gotten the Lagrangians (14a), (15a) and (16a). However, the analytical functions appearing in these expression represent a real challenge for the quantization with the path integration method. One must also note from relations (14a) and (15a) that these can not be expressed in a covariant way since Lorentz transformations do not leave invariant this dissipation system.

3. QUANTIZATION OF THE CONSTANT OF MOTION

We are interested here in the quantization of the system (1) at first order in the dissipation parameter β, characterized by the constant of motion (11a). As it was mentioned above, the quantization using the Hamiltonian or the Lagrangian approaches does not look plausible due to the implicit form of the Hamiltonian and the complicated expression for the Lagrangian. However, one can try to use the idea of quantizing the velocity (López, 2000) through the obvious expression

$$\widehat{v} = -i\frac{\hbar}{m}\frac{\partial}{\partial x}. \tag{18}$$

In this way, one can use directly the constant of motion of our system to make its quantization through the equivalent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{K}(\widehat{x}, \widehat{v})\Psi, \tag{19}$$

where $\widehat{K}$ is the Hermitian linear operator associated to the constant of motion K and which has units of energy.

Thus, let us consider the constant of motion (11a). From (19) and (11a), the equation obtained is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \left\{ \frac{mc^2}{\sqrt{1 - \widehat{v}^2/c^2}} - m\beta c^3 \left[\frac{\widehat{v}}{\sqrt{1 - \widehat{v}^2/c^2}} - \arcsin \frac{\widehat{v}}{c} \right] + fx \right\} \Psi. \tag{20}$$

Equation (20) represents an stationary problem, and the proposition

$$\Psi(x, t) = \psi(x) \exp\left(-i\frac{E\,t}{\hbar}\right) \tag{21}$$

transforms (20) to an eigenvalue problem,

$$\left\{\frac{mc^2}{\sqrt{1 - \widehat{v}^2/c^2}} - m\beta c^3\left[\frac{\widehat{v}}{\sqrt{1 - \widehat{v}^2/c^2}} - \arcsin\frac{\widehat{v}}{c}\right] + fx\right\}\psi = E\psi. \tag{22}$$

One can see from this expression that it is better to look for its solution in the velocity representation, which is given by applying the Fourier transformation to the function ψ,

$$\phi(v) = F[\psi(x)] = \frac{1}{\sqrt{2\pi}}\int_R e^{imvx/\hbar}\psi(x)dx, \tag{23}$$

where the variable v represents the velocity of the particle. Applying this Fourier transformation to (22), one gets

$$\left\{\frac{1mc^2}{\sqrt{1 - v^2/c^2}} - m\beta c^3\left[\frac{v}{\sqrt{1 - v^2/c^2}} - \arcsin\frac{v}{c}\right] + i\frac{\hbar f}{m}\frac{\partial}{\partial v}\right\}\phi = E\phi. \tag{24}$$

This equation has the following solution

$$\phi_E(v) = \frac{1}{2v_o}e^{i\frac{m^2c^3}{\hbar f}\left[\arcsin\frac{v}{c} - 2\beta c\sqrt{1 - \frac{v^2}{c^2}} - \beta v\arcsin\frac{v}{c} - \frac{Ev}{mc^3}\right]}, \tag{25}$$

where v_o represent some maximum velocity of the particle ($v_o \le c$), and $[-v_o, v_o]$ is the intervale of velocities where the normalization of the function (25) has been carried out,

$$\int_{-v_o}^{v_o}|\phi_E(v)|^2dv = 1. \tag{26}$$

One also has that

$$\langle\phi_E|\phi_{E'}\rangle = \int \phi_E^*(v)\phi_{E'}(v)dv = \frac{\hbar f}{m}\delta(E - E'). \tag{27}$$

The spectrum of energies of the particle is continuous because of the form of the potential, fx, and the general solution of (20) can be written, using (21) and (25), as

$$\Phi(v, t) = \int A(E)\phi_E(v)e^{-iEt/h}dE, \tag{28}$$

where the coefficient $A(E)$ is determinate by the initial condition $\Phi(v, 0)$ in the following way

$$A(E) = \frac{m}{\hbar f} \int \phi_E^*(v)\Phi(v, 0)dv .\tag{29}$$

Now, in the case the potential be of the form

$$V(x) = \begin{cases} fx \text{ if } x > 0 \\ \\ \infty \text{ if } \leq 0 \end{cases}\tag{30}$$

which would correspond to the one-dimensional dissipative relativistic bouncer problem, one requires that $\psi(0) = 0$ and $\psi(x) = 0$ if $x < 0$ for the solution of (22). This condition brings about the discrete spectrum of the system and can be written in the velocity representation, using the inverse Fourier transformation, in the following way

$$0 = \psi(0) = F^{-1}[\phi_E(v)]\Big|_{x=0} = \frac{1}{\sqrt{2\pi}} \int \phi_E(v)dv .\tag{31}$$

Given the set of eigenvalues, $\{E_n\}$, the eigenfunctions are given by

$$\phi_n(v) = \frac{1}{\sqrt{2v_o}} e^{i\frac{m^2c^3}{\hbar f}\left[\arcsin\frac{v}{c} - 2\beta c\sqrt{1-\frac{v^2}{c^2}} - \beta v \arcsin\frac{v}{c} - \frac{E_n v}{mc^3}\right]},\tag{32}$$

and the general solution can be written as

$$\Phi(v, t) \sum_n A_n \Phi_n(v)e^{-iE_n t/h},\tag{33}$$

where, using the orthogonality $\langle\phi_n|\phi_{n'}\rangle = \delta_{n,n'}$ the coefficients A_n's are determinate by the initial condition $\Phi(v, 0)$ and through the following expression

$$A_n = \int_{-v_o}^{v_o} \phi_n^*(v)\Phi(v, 0)\, dv .\tag{34}$$

The wave function in the x representation is gotten through the inverse Fourier transformation,

$$\psi_n(x) = \frac{1}{\sqrt{2\pi}} \int e^{-imvx/\hbar}\phi_n(v)\, dv .\tag{35}$$

Note that the above quantization process can be done in general for the constant of motion (9) and any case defined by (6).

4. CONCLUSION

For a relativistic particle under a constant force and a dissipative force proportional to the velocity, a local constant of motion has been found, and one has

outlined the problem to get its Lagrangian and Hamiltonian in general. To quantize this system, Schrödinger quantization approach has been used with this local constant of motion at first order on the dissipation parameter and using the velocity representation of the wave function. Using this quantization approach, we have outlined the dissipative relativistic quantum bouncer problem.

APPENDIX

The function $B(v)$ is given by

$$
B(v) = \begin{cases}
\frac{mc^2}{1-\beta^2 c^2}\left[-\sqrt{1-\frac{v^2}{c^2}} + \beta v \log \frac{1+\sqrt{1-\frac{v^2}{c^2}}}{v/c} \right] + \frac{m\beta c^3 f_1(v)}{(1-\beta^2 c^2)^{3/2}} & \text{if } \beta c < 1 \\[3ex]
-mcv \log\left(\frac{2(1+\sqrt{1-v^2/c^2})}{v/c} \right) + \frac{mc^2 f_2(v)}{3} & \text{if } \beta c = 1 \\[3ex]
\frac{mc^2}{1-\beta^2 c^2}\left[-\sqrt{1-\frac{v^2}{c^2}} + \beta v \log \frac{1+\sqrt{1-\frac{v^2}{c^2}}}{v/c} \right] \\[2ex]
+\frac{m\beta c^3 f_3(v)}{(1-\beta^2 c^2)^{3/2}} & \text{if } \beta c > 1
\end{cases} \tag{A.1}
$$

where f_1, f_2 and f_2 are defined as

$$
f_1(v) = v \int \frac{\arcsin\left(\frac{\beta c + v/c}{1+\beta v} \right) dx\, v}{v^2}, \tag{α_1}
$$

$$
f_2(v) = \frac{-1 - \frac{v}{c} + \frac{v^2}{c^2} + \frac{3v}{c}\sqrt{1-\frac{v^2}{c^2}} \log\left(\frac{2(1+\sqrt{1-v^2/c^2})}{v/c} \right)}{\sqrt{1-v^2/c^2}}, \tag{α_2}
$$

and

$$
f_3(v) = v \int \frac{\log\left(\frac{\psi(v)}{1+\beta v} \right) dv}{v^2}, \tag{α_3}
$$

where the function ψ is given by (7b). The function $C(v)$ is given by

$$
C(v) = \begin{cases}
\dfrac{mc^2}{1-\beta^2 c^2}\left[\dfrac{v-\beta c^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} + \beta \log\left(\dfrac{1+\sqrt{1-v^2/c^2}}{v/c}\right)\right] \\[2ex]
+\dfrac{m\beta^3 g_1(v)}{(1-\beta^2 c^2)^{3/2}} & \text{if } \beta c < 1 \\[3ex]
-mc\left[\log\left(\dfrac{2(1+\sqrt{1-v^2/c^2})}{v/c}\right) - \dfrac{1}{\sqrt{1-v^2/c^2}}\right] \\[2ex]
+\dfrac{mcg_2(v)}{3} & \text{if } \beta c = 1 \\[3ex]
\dfrac{mc^2}{1-\beta^2 c^2}\left[\dfrac{v-\beta c^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} + \beta \log\left(\dfrac{1+\sqrt{1-v^2/c^2}}{v/c}\right)\right] \\[2ex]
+\dfrac{m\beta^3 g_3(v)}{(1-\beta^2 c^2)^{3/2}} & \text{if } \beta c > 1
\end{cases}
\tag{A.2}
$$

where g_1, g_2 and g_3 are defined as

$$
g_1(v) = \frac{f_1(v)}{v} + \frac{\arcsin\left(\frac{\beta c + v/c}{1+\beta v}\right)}{v} \tag{β_1}
$$

$$
g_2(v) = \frac{g_{21}(v) + g_{22}(v)}{(1-v^2/c^2)^{3/2}(1+\sqrt{1-v^2/c^2})}, \tag{β_2}
$$

and

$$
0g_3(v) = \frac{f_3(v)}{v} + \frac{1}{v}\log\left(\frac{\psi(v)}{1+\beta v}\right). \tag{β_3}
$$

The functions g_{21} and g_{22} have been defined as

$$
g_{21}(v) = (-4 - \frac{v}{c} - \frac{3v^2}{c^2} + \frac{v^3}{c^3})(1 + \sqrt{1-\frac{v^2}{c^2}}) \tag{δ_1}
$$

and

$$
g_{22}(v) = 3(1-\frac{v^2}{c^2})\left[-\frac{v^2}{c^2} + 1 + \sqrt{1-\frac{v^2}{c^2}}\right]\log\left(\frac{2(1+\sqrt{1-v^2/c^2})}{v/c}\right) \tag{δ_2}
$$

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